Feb. 16

7.1.9. If $f \in \mathcal{R}[a,b]$, and if $(\dot{P}_n)$ is any sequence of tagged partions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$, prove that $\int_a^b f = \lim_n S(f; \dot{P}_n)$.

Proof. $(f \in \mathcal{R}[a,b])$

$$\Longleftrightarrow \left(\begin{array}{l} \exists L \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0, \\ \forall \dot{P} \text{ tagged partition},\ \|\dot{P}\| < \delta_\varepsilon \\ \quad \Rightarrow \quad |S(f;\dot{P}) - L| < \varepsilon \end{array} \right) \quad ①$$

$$(\|\dot{P}_n\| \longrightarrow 0)$$

$$\Longleftrightarrow (\forall \delta_\varepsilon > 0,\ \exists N_{\delta_\varepsilon} \in \mathbb{N},\ \forall n > N_{\delta_\varepsilon},\ \|\dot{P}_n\| < \delta_\varepsilon.) \quad ②$$

$$①, ② \Rightarrow \forall \varepsilon > 0,\ \exists N_{\delta_\varepsilon} \in \mathbb{N},\ \forall n > N_{\delta_\varepsilon},$$
$$|S(f, \dot{P}_n) - L| < \varepsilon \Longleftrightarrow \lim_{n \to \infty} S(f, \dot{P}_n) = L =: \int_a^b f.$$

$\blacksquare$

Remark.

$$\left(\lim_{x \to a} f(x) = L\right) \Longleftrightarrow \left(\forall \{x_n\}_{n \in \mathbb{N}},\ \lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = a\right)$$

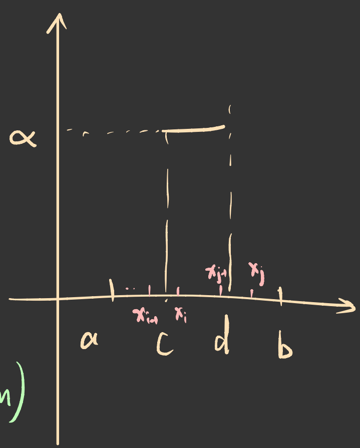
<u>7.1.13</u> Suppose that $c \le d$ are points in $[a,b]$. If $\varphi : [a,b] \to \mathbb{R}$ satisfies $\varphi(x) = \alpha > 0$ for $x \in [c,d]$ and $\varphi(x) = 0$ elsewhere in $[a,b]$, Prove that $\varphi \in \mathcal{R}[a,b]$ and that $\int_a^b \varphi = \alpha(d-c)$.

<u>Proof</u>. May assume $c<d$, or Theorem 7.1.3 implies.

$\forall \varepsilon > 0$, let $\delta_\varepsilon = \frac{\varepsilon}{4\alpha}$, we may assume $\frac{\varepsilon}{4\alpha} \ll d-c$.

If $\dot{P}$ is a tagged partition such that $\|\dot{P}\| < \delta_\varepsilon$

$$\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$$



$\exists\, i$, $x_{i-1} \le c < x_i$

$\exists\, j$, $x_{j-1} < d \le x_j$ ($j-1 > i$ by assumption)

$$\Rightarrow \quad f(t_k) = \begin{cases} \alpha, & i+1 \le k \le j-1; \\ 0 \text{ or } \alpha, & k = i, j; \\ 0, & \text{otherwise.} \end{cases}$$

$$S(\varphi, \dot{P}) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \sum_{i \le k \le j} f(t_k)(x_k - x_{k-1})$$

① If $f(t_i) = f(t_j) = \alpha$, $S(\varphi, \dot{P}) = \sum_{i \le k \le j} \alpha(x_k - x_{k-1}) = \alpha(x_j - x_{i-1})$

$$S(\varphi,\dot{P})-\alpha(d-c)\le\alpha(x_j-x_{i-1})-\alpha(d-c)=\alpha\big((x_j-d)+(c-x_{i-1})\big)$$
$$\le\alpha\big((x_j-x_{j-1})+(x_i-x_{i-1})\big)\le\alpha\left(\frac{\varepsilon}{4\alpha}+\frac{\varepsilon}{4\alpha}\right)=\frac{\varepsilon}{2}$$

② If $f(t_i)=f(t_j)=0$, $S(\varphi,\dot{P})=\sum\limits_{i+1\le k\le j-1}\alpha(x_k-x_{k-1})=\alpha(x_{j-1}-x_i)$

$$S(\varphi,\dot{P})-\alpha(d-c)\ge\alpha(x_{j-1}-x_i)-\alpha(d-c)$$
$$=-\alpha\big((d-x_{j-1})+(x_i-c)\big)\ge-\alpha\big((x_j-x_{j-1})+(x_i-x_{i-1})\big)$$
$$\ge-\alpha\left(\frac{\varepsilon}{4\alpha}+\frac{\varepsilon}{4\alpha}\right)=-\frac{\varepsilon}{2}.$$

In conclusion, $|S(\varphi,\dot{P})-\alpha(d-c)|<\frac{\varepsilon}{2}<\varepsilon.$ $\boxed{\text{III}}$

7.1.14. Let $0 \le a \le b$, $Q(x) = x^2$ for $x \in [a,b]$ and let $P := \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a partition of $[a,b]$.

For each i, let q_i be the positive square of

$$\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2).$$

a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.

b) Show that $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

c) If $\dot{Q}$ is the tagged partition with the same subintervals as P and the tags q_i. Show that

$$S(Q, \dot{Q}) = \frac{1}{3}(b^3 - a^3)$$

d) Use the argument in Example 7.1.4(c) to show that $Q \in R[a,b]$ and $\int_a^b Q = \int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$.

Proof. a) $q_i = \sqrt{\frac{1}{3}(x_i^2+x_ix_{i-1}+x_{i-1}^2)} \geq \sqrt{\frac{1}{3}(x_{i-1}^2+x_{i-1}^2+x_{i-1}^2} = x_{i-1} \geq 0$

$q_i \leq \sqrt{\frac{1}{3}(x_i^2+x_i^2+x_i^2} \leq x_i$

b) $Q(q_i)(x_i-x_{i-1}) = \frac{1}{3}(x_i^2+x_ix_{i-1}+x_{i-1})(x_i-x_{i-1})$

$= \frac{1}{3}x_i^3 \left(1+\frac{x_{i-1}}{x_i}+\left(\frac{x_{i-1}}{x_i}\right)^2\right)\left(1-\frac{x_{i-1}}{x_i}\right)$

$= \frac{1}{3}x_i^3 \frac{1-\left(\frac{x_{i-1}}{x_i}\right)^3}{1-\frac{x_{i-1}}{x_i}}\left(1-\frac{x_{i-1}}{x_i}\right) = \frac{1}{3}(x_i^3-x_{i-1}^3)$

$\left(1+q+q^2+\cdots+q^n = \frac{1-q^{n+1}}{1-q}\right)$

c) $S(Q,\dot{Q}) = \sum_{i=1}^{n} Q(q_i)(x_i-x_{i-1})$

$= \sum_{i=1}^{n} \frac{1}{3}(x_i^3-x_{i-1}^3) = \frac{x_n^3-x_0^3}{3} = \frac{b^3-a^3}{3}$

d) let $\dot{P} := \{[x_{i-1},x_i], t_i\}$ be a tagged partition of $[a,b]$.
Assume $\|\dot{P}\| \leq \delta$. $\dot{Q} := \{[x_{i-1},x_i], q_i\}$ q_i defined as above.

$|S(Q,\dot{Q})-S(Q,\dot{P})| = \left|\sum_{i=1}^{n} t_i^2(x_i-x_{i-1}) - \sum_{i=1}^{1} q_i^2(x_i-x_{i-1})\right|$

$t_i^2-q_i^2 \leq x_i^2-x_{i-1}^2 \leq (x_i-x_{i-1})(x_i+x_{i-1}) \leq \delta\cdot 2b$

$t_i^2-q_i^2 \geq x_{i-1}^2-x_i^2 \geq -2\delta b$

$\Rightarrow |S(Q,\dot{Q})-S(Q,\dot{P})| \leq \sum_{i=1}^{n}|t_i^2-q_i^2|(x_i-x_{i-1}) \leq 2\delta b\sum_{i=1}^{n}(x_i-x_{i-1})$

$= 2\delta b(b-a)$

$\forall \varepsilon < 0, \ \exists \ \delta_\varepsilon = \dfrac{\varepsilon}{2b(b-a)}$

$\forall \dot{P}$ s.t. $\|\dot{P}\| < \delta_\varepsilon$.

$$\left| S(Q, \dot{P}) - \frac{1}{3}(b^3 - a^3) \right| = \left| S(Q, \dot{P}) - S(Q, \dot{Q}) \right| \leq 2\delta b(b-a) = \varepsilon.$$